SOLUTION OF A BOUNDARY - VALUE PROBLEM BY PARTIAL DIFFERENTIAL EQUATIONS AND FOURIER SERIES

A THESIS

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CHAPTER I

INTRODUCTION

From an applied point of view, perhaps the most important manner in which partial differential equations may arise is from a mathematical formulation of physical problems. Ordinary differential equations have the number of variables involved so restricted in number (one dependent and one independent variable) that they can describe only relatively simple applied problems. Partial differential equations, on the other hand, can have any number of independent variables, and so we might expect to be able to describe more complex applied problems.

In many instances the formulation of a partial differential equation from a consideration of the physical situation is easy. In our case, which is to find a solution, subject to certain initial or boundary conditions, it is difficult to satisfy the associated conditions. This is in contrast to ordinary differential equations where once the (general) solution has been obtained it is usually possible to satisfy the conditions. As in the case of ordinary differential equations, the problem of determining a solution of a partial differential equation subject to certain conditions is often called a boundary-value problem.

In other instances, the mathematical formulation of a physical problem is very difficult. Many problems now remain unsolved because of such difficulty. In our case, crude approximations to reality will be made in order to produce a formulation, the hope being that any errors thus arising will be small.

We shall show how a partial differential equation, subject to boundary conditions, arises from a physical problem. After we have shown how it arises, we will solve it, using Fourier series.

CHAPTER II

DEFINITIONS AND SYMBOLS

Definition of Terms.--For the purpose of clarity, certain terms used in this thesis have been defined as follows:

1. Boundary - Value Problem - The problem of determining solutions of an ordinary or partial differential equation subject to conditions is called a boundary-value problem.

2. Force - The time rate of change of momentum is called force.

3. Acceleration - The rate of change in velocity with respect to time is called acceleration.

4. Density - Mass per unit volume is called density.

5. Volume - The total space occupied by matter is called volume.

6. Mass - The quantity of matter that a body possesses is called mass.

7. Differential Equation - Is an equation which involves derivatives of a dependent variable with respect to one or more independent variables.

8. Ordinary Differential Equation - A differential equation which involves derivatives with respect to a single independent variable is called an ordinary differential equation.

9. Partial Differential Equation - A differential equation which involves derivatives with respect to two or more independent variables is called a partial differential equation.

10. General Solution - A differential equation of order n which contains n arbitrary constants is a general solution.

11. Particular Solution - A solution of a differential equation which can be obtained from the general solution by giving special values to the arbitrary constants is called a particular solution.

12. Singular Solution - A solution of a differential equation which cannot be obtained from the general solution by any choice of the arbitrary constants is called a singular solution.

13. Mathematical Formulation of a Scientific Problem - Scientific laws, which are of course based on experiment, are translated

into a mathematical equation or equations is called a mathematical formulation of a scientific problem.

14. Linear Momentum - The linear momentum of a body is the product of its mass m by its velocity v.

15. Linear Impulse - The linear impulse is the product of the force F and the time t that the force acts.

16. Velocity - The rate of change in distance with respect to time is called velocity.

17. Period - The time for one complete cycle is called the period.

18. Frequency - The number of cycles per second is called the frequency.

19. Fourier Series - A function with a constant term, an infinite number of sine terms, and an infinite number of cosine terms is called a Fourier series.

Symbols.--For the purpose of clarity, certain symbols used in this thesis have been defined as follows:

20. Forevery . . . . $\forall$

21. Acceleration . . . . $\frac{\partial^2 y}{\partial t^2}$

22. Velocity . . . . $\frac{\partial y}{\partial t}, y_t$

23. Tan θ . . . . $\frac{\partial y}{\partial x}$

24. Function of X and t . . . . $F(X,t)$

25. Density . . . . ρ

26. Tension . . . . T

27. Horizontal Force . . . . F_h

28. Vertical Force . . . . F_v

29. There Exists . . . . $\exists$

30. Natural Logarithms Base . . . . e

31. Therefore . . . . $\therefore$

32. That is to Say . . . . i.e.

33. Summation . . . . $\sum$

34. Implication . . . . $\Longrightarrow$

CHAPTER III

MATHEMATICAL FORMULATION OF A BOUNDARY - VALUE PROBLEM

<u>The Problem of the Vibrating String</u>.—A violin string is tightly stretched between two fixed points $x = o$ and $x = L$ on the horizontal axis of a violin of Figure 1. At time $t = o$ the string is picked up at the middle (Figure 2), to a distance h. Then the string is released. Describe the motion which takes place.

Clearly many things could happen. The string could be so tightly stretched that when we lifted the middle a height h the string would break. This case is simple and we shall not consider it. It is more natural to assume that the string is perfectly flexible and elastic. Also, to simplify the problem, we assume that h is small compared with L. Other assumptions will be made as we proceed.

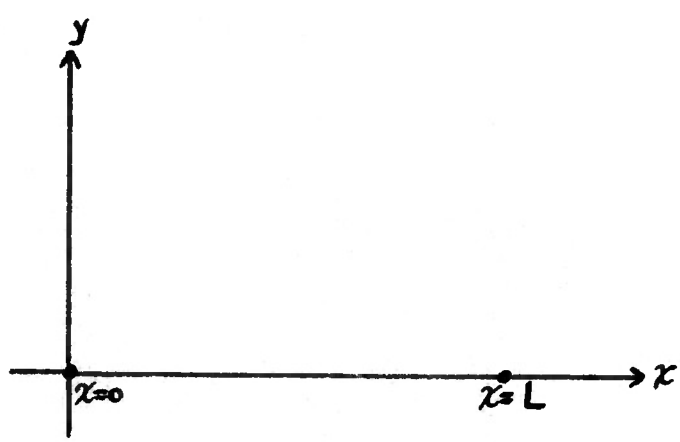

Figure 1

Position of string before it is picked up.

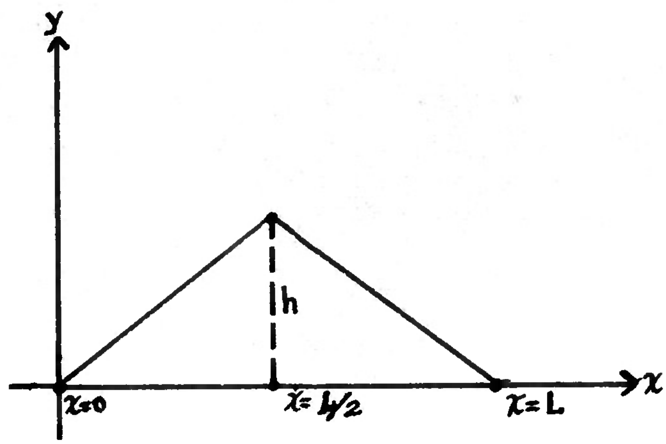

Figure 2

Position of string after it is picked up to a height h in the middle.

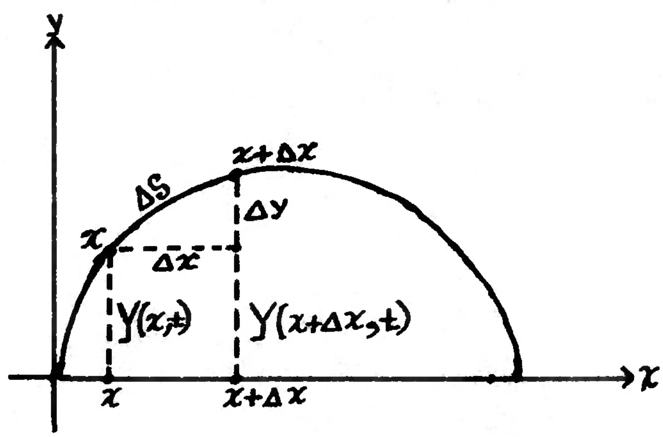

Figure 3

Equilibrium Position

Drop the only perpendicular from point x to a point on the line x + Δx. Call it Δx and the portion of x + Δx from the perpendicular to the point x + Δx call its increment Δy. Now the portion of the string between points x and x + Δx on the curve is an increment Δs.

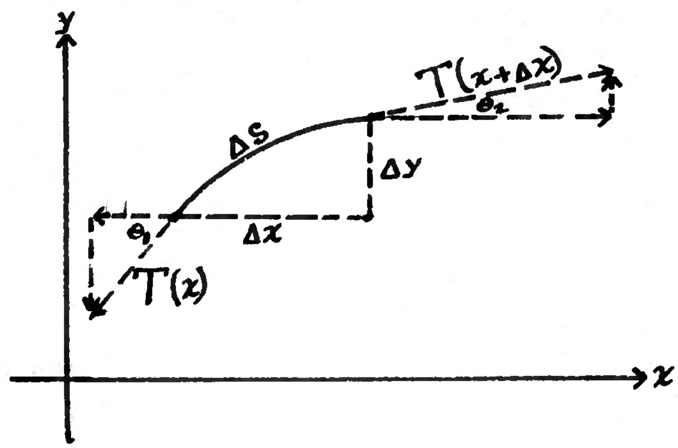

Figure 4

Mathematical Formulation.--Let us suppose that at some instant , the string has a shape as shown in Figure 3. We shall call Y(x,t) the displacement of point x on the string (measured from the equilibrium position which we take as the x axis) at the time t. The displacement, at time t, of the neighboring point x + Δx will then be given by Y(x + Δx,t).

In order to describe the motion which ensues, we consider the forces acting on the small element of the string between x and x + Δx shown considerably enlarged in Figure 4. There will be two forces acting on the element, the tension T(x) due to the portion of the string to the left, and tension T(x+ Δx) due to the portion to the right. Note that we have for the moment assumed that the tension depends on position. Resolving these forces into components gives:

Forces acting on point $x + \Delta x$.

$F_h = T(x + \Delta x) \cos \theta_2$.

$F_v = T(X + \Delta x) \sin \theta_2$.

Forces acting on point x.

$F_h = -T(x) \cos \theta_1$.

$F_v = -T(x) \sin \theta_1$.

$$\sum F_v = T(x + \Delta x) \sin\theta_2 - T(x) \sin\theta_1. \qquad (1)$$

$$\sum F_h = T(x + \Delta x) \cos\theta_2 - T(x) \cos\theta_1.$$

We now assume that there is no right and left motion of the string, i.e., to a high degree of approximation the net horizontal force is zero. This agrees with the physical situation. The net vertical force in (1) produces an acceleration of the element. Assuming the string has density (mass per unit length) ρ, the mass of the element is $\rho \Delta S$. The vertical acceleration of the string is given approximately by $\frac{\partial^2 y}{\partial t^2} + \epsilon$ where $\epsilon \rightarrow 0$ as $\Delta x \rightarrow 0$. Newton's Second Law of motion states that the rate of change in momentum of a body is proportional to the net force acting on the body and has the same direction as the force. Since momentum equals impulse we can write Ft = mv, where F is force, t is time, m is mass and v is velocity.

$$F\,t = m\,v$$

$$F\,t = m\frac{\partial y}{\partial t} \qquad (a)$$

Differentiating both sides of equation (a) we get

$$F = m\,\frac{\partial^2 y}{\partial t^2}. \qquad (a^1)$$

Hence,

$$T(x + \Delta x)\sin\theta_2 - T(x)\sin\theta_1 = \rho\Delta s\frac{\partial^2 y}{\partial t^2} \qquad (2)$$

to a high degree of accuracy. If θ is the angle which the tangent at any point of the element makes with the positive X axis, then θ is a function of position and we write $\theta_1 = \theta(x)$, $\theta_2 = \theta(x + \Delta x)$.

Substitution into (2) we get

$$T(x + \Delta x)\sin\theta(x + \Delta x) - T(x)\sin\theta(x) = \rho\Delta s\frac{\partial^2 y}{\partial t^2}.$$

Dividing by Δx yields

$$\frac{T(x + \Delta x)\sin\theta(x + \Delta x) - T(x)\sin\theta(x)}{\Delta x} = \rho\frac{\Delta s}{\Delta x}\cdot\frac{\partial^2 y}{\partial t^2}. \qquad (3)$$

We now make the assumption that only those motions will be considered for which the slope of the tangent at any point on the string is small, i.e. the degree of smallness depends, of course, on accuracy desired. It follows that $\Delta s/\Delta x$ is very close to unity and that $\sin\theta(x)$ and $\sin\theta(x + \Delta x)$ can be replaced with high accuracy by $\tan\theta(x)$ and $\tan\theta(x + \Delta x)$, respectively. This is certainly true since the $\sin\theta$ for a very small θ is approximately equal to the $\tan\theta$. To see this clearly, expand $\sin\theta$ in a Maclaurin series.

Let $F(\theta) = \sin\theta$

$F(\theta) = \sin\theta$	$F(o) = o$
$F^1(\theta) = \cos\theta$	$F^1(o) = 1$
$F^{11}(\theta) = -\sin\theta$	$F^{11}(o) = o$
$F^{111}(\theta) = -\cos\theta$	$F^{111}(o) = -1$

The required Maclaurin series is

$$F(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \text{--------} \quad (\forall\text{ real value of }\theta). \qquad (b)$$

Now for a small θ, $F(\theta) = \theta$ since the other part of the series will be very small in relationship to the term θ. In view of these approximations we may write (3) as

$$\frac{T(x+\Delta x)\tan\theta(x+\Delta x) - T(x)\tan\theta(x)}{\Delta x} = \rho\frac{\partial^2 y}{\partial t^2} \qquad (4)$$

to a high degree of accuracy. In the limit as $\Delta x \to o$, (4) becomes

$$\frac{\partial}{\partial x}\left[T(x)\tan\theta(x)\right] = \rho\frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial}{\partial x}\left[T(x)\frac{\partial y}{\partial x}\right] = \rho\frac{\partial^2 y}{\partial t^2} \qquad (5)$$

since $\tan\theta(x) = \frac{\partial y}{\partial x}$. Equation (5) is called vibrating string equation or more generally the wave equation.

If $T(x) = T$, a constant, the equation can be written

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2} \qquad (6)$$

where $a^2 \equiv \frac{T}{\rho}$. It is assumed that the tension is constant.

Let us now see what the boundary conditions are. Since the string is fixed at points x=o and x=L, we have

$$Y(o,t) = o\ ,\ Y(L,t) = 0 \text{ for } t \geq 0. \qquad (7)$$

These state that the displacements at the ends of the string are always zero. Referring to Figure 2, and using the point slope formula, it is seen that

$$\frac{y-y_1}{x-x_1} = \frac{y-y_2}{x-x_2}$$

$$\frac{y-0}{x-0} = \frac{h-0}{\frac{L}{2}-0} \qquad y = \frac{2h}{L}x, \text{ for the left half of the string.}$$

$$\frac{y-0}{x-L} = \frac{h-0}{\frac{L}{2}-L} \qquad y = -\frac{2h}{L}(L-x), \text{ for the right half of the string.}$$

$$y(x,o) = \begin{cases} \frac{2hx}{L} & 0 \leq x \leq \frac{L}{2} \\ \frac{2h}{L}(L-x) & \frac{L}{2} \leq x \leq L \end{cases} \qquad (8)$$

This merely gives the equations of the two straight line portions of that figure. $Y(x,0)$ denotes the displacement of any point x at t=0. Since the string is released from rest, its initial velocity everywhere is zero. Denoting by y_t the velocity $\frac{\partial y}{\partial t}$ we may write

$$Y_t(x,0) = 0 \tag{9}$$

which says that the velocity at any place x at time t=0 is zero.

The differential equation (6) together with the boundary conditions (7), (8), and (9) constitute our boundary - value problem.

CHAPTER IV

SOLUTION OF A LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Let a solution of (6) be of the form $Y(x,t) = XT$, where X is a real function of x alone, and T is a real function of t alone. (10)

$$\frac{\partial Y}{\partial t}(x,t) = XT^1 \qquad \frac{\partial Y}{\partial x}(x,t) = TX^1$$

$$\frac{\partial^2 Y}{\partial t^2}(x,t) = XT'' \quad (11) \qquad \frac{\partial^2 Y}{\partial x^2}(x,t) = TX'' \quad (12)$$

Substitution of (11) and (12) in (6) yields

$$XT'' = a^2TX''.$$

Separating the variables, we have

$$\frac{X''}{X} = \frac{T''}{a^2T} \mp C. \qquad (13)$$

Since one side of (13) depends on x, while the other depends on t, it follows that each side is constant. Calling this constant c and considering the cases $C < 0$, $c = 0$, $c > 0$, we may show that only $c < 0$ yields anything. This can be seen physically by observing that as $t \to \infty$ the stretch of the string becomes unbounded if $c > 0$, this violates a fundamental physical fact. If $c = 0$, the stretch of the string is independent of time. Hence, we assume that $c = -\lambda^2$ and obtain from (13)

$$\frac{X''}{X} = -\lambda^2 \qquad \text{and } \frac{T''}{a^2T} = -\lambda^2$$

$X'' + \lambda^2 X = 0$ (14) $\qquad$ $T'' + \lambda^2 a^2 T = 0$ (15)

Equations (14) and (15) are linear homogeneous differential equations with constant coefficients. We must find a way to solve this type of equation. To find a method of solving the above equations it is necessary to know the following information:

1. Does e exist?

$$\text{i.e. } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Proof: let $b_n = (1 + \frac{1}{n})^n$

Expand $(1 + \frac{1}{n})^n$ using the Bionomial theorem.

$$b_n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \ldots \frac{1}{k!}(1 - \frac{1}{n})(1 - \frac{2}{n})\ldots(1 - \frac{k-1}{n}) +$$

$$\ldots + \frac{1}{n!}(1 - \frac{1}{n}) \ldots (1 - \frac{n-1}{n})$$

$$b_n < 1 + 1 + \frac{1}{2!} + \ldots\ldots + \frac{1}{k!} + \ldots\ldots + \frac{1}{n!} < e^1$$

Hence b_n is bounded above by e^1.

It is also clear that the kth term of the expansion for b_n is smaller than that for b_n+1:

$$\frac{1}{k!}(1 - \frac{1}{n})(1 - \frac{2}{n})\ldots(1 - \frac{k-1}{n}) < \frac{1}{k!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1})\ldots(1 - \frac{k-1}{n+1}).$$

This is true because

$(1 - \frac{j}{n}) < (1 - \frac{j}{n+1})$ $\quad j=1,2,\ldots\ldots(k-1) \leq n-1$, and b_n+1

contains one more positive term than b_n.

Hence $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$ exists and

$$e \leq e^1,$$

since e' is an upper bound for b_n and e is the least upper bound for b_n.

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e. \tag{C}$$

We have seen that when n is very large $e = \left(1+\frac{1}{n}\right)^n$, therefore, when n is still very large,

$e^x = \left[\left(1+\frac{1}{n}\right)^n\right]^x = \left(1+\frac{1}{n}\right)^{nx}$. Expressed in symbols the full statement is

$$e^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} \tag{C^1}$$

and we can expand $\left(1+\frac{1}{n}\right)^{nx}$ in a series formula, each term of which can be computed when n is infinite.

Using the binomial expansion, we get,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\frac{1}{n} + \frac{nx(nx-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{nx(nx-1)(nx-2)}{3!}\frac{(1)^3}{n^3} + \text{-----}$$

and again taking out n as a factor in each of the parentheses,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\frac{1}{n} + \frac{n^2x\left(x-\frac{1}{n}\right)}{2!\,n^2} + \frac{n^3x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!\,n^3} + \text{----}$$

$$= 1 + x + \frac{\left(x\left(x-\frac{1}{n}\right)\right)}{2!} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{3!} + \text{--------------},$$

To find the limit which this expression approaches when n is infinite, we again suppose that in each of the fractions $\frac{1}{n}$, $\frac{2}{n}$, etc., n is infinites; each fraction then becomes zero and each term in parentheses is simply x. The limit of the entire series is then,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx} = 1 + x + \frac{x.x}{2!} + \frac{x.x.x.}{3!} + \frac{x.x.x.x.}{4!} + \text{------},$$

and hence, according to (C^1),

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{----} \frac{x^n}{n!} + \text{----} (\forall \text{ real value of } x) \quad (d)$$

2. Eule's formula: $e^{ix} = \cos x + i \sin x$

Proof:

Using (d) we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{-----} \frac{x^n}{n!} + \text{-----}.$$

Let $x = i\,x$.

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \cdots$$

$$e^{ix} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{-----------}\right).$$

Now the $\cos\theta$ can be expanded as a Maclaurin series.

Let $F(\theta) = \cos\theta$

$F(\theta) = \cos\theta$	$F(o) = 1$
$F^1(\theta) = -\text{sine}\,\theta$	$F'(o) = o$
$F''(\theta) = -\cos\theta$	$F''(o) = -1$
$F'''(\theta) = \sin\theta$	$F'''(o) = o$
$F^{iv}(\theta) = \cos\theta$	$F^{iv}(o) = 1$

$$F(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (e)$$

By (b) and (e):

$$e^{ix} = \cos x + i \sin x. \quad (e')$$

3. A Method of Solving a Linear Homogeneous Differential Equation With

Constant Coefficients.

A linear homogenous differential equation with constant coefficients of order n, defining y as a function of x, can be written in the form

$$\frac{d^n y}{dx^n} + \frac{a_1 d^{n-1}}{dx^{n-1}} y + \frac{a_2 d^{n-2}}{dx^{n-2}} y + \text{-----} \frac{a_{n-1} dy}{dx} + a_n y = 0,$$

where a_1, a_2, ----, a_n are functions of x alone.

The characteristic equation with Real roots.

Consider $\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = o$ or $F(D)y=o$,

where $D = \frac{dy}{dx}$ and $D^2 + a_1 D + a_1 O + a_2$, in which a_1 and a_2 are constants, Since the linear equation of the first order $y' - my = o$ has the solution $y = ce^{mx}$, it is our hope to find similar solutions for (F).

Theorem: A function $y = e^{mx}$ satisfied $F(D)y=o$ if and only if $F(m) = o$ or

$$m^2 + a_1 m + a_2 = o.$$

Proof: To obtain m so that $y = e^{mx}$ satisfied (F), we substitute $y = e^{mx}$, $y' = me^{mx}$, and $y'' = m^2 e^{mx}$ in (F), and find

$$m^2 e^{mx} + a_1 m e^{mx} + a_2 e^{mx} = o, \text{ or } e^{mx}(m^2+a_1 m+a_2) = o. \quad (F')$$

Since $e^{mx} \neq o$ for any values of a m and x, from (F') we obtain

$$m^2 + a_1 m + a_2 = o.$$

We shall call (F'') the characteristic equation, or the auxiliary equation, for (F). The roots of (F'') may be real and distinct, real and equal, or imaginary.

If the characteristic equation has distinct real roots $m = m_1$ and $m = m_2$, then (F) has the linearly independent solutions

$y = e^{m_1 x}$ and $y = e^{m_2 x}$, and the general solution

$$y = C_1 e^{m_1 x} + C_2 e^{m_2 x},$$

where C_1 and C_2 are arbitrary constants.

The characteristic equation with imaginary roots. Consider the equation of the second order

$$(D^2 + a_1 D + a_2)\, y = o, \text{ or } F(D)y = o, \qquad (g)$$

when the characteristic equation $F(m) = o$ has imaginary roots $m = \alpha \pm \beta i$, where α and β are real and $\beta \neq o$.

Then, the general solution of (g) is

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x} = e^{\alpha x}(C_1 e^{i\beta x} + C_2 e^{i\beta x}). \qquad (g')$$

In (g') use Euler's formula (e') to obtain

$$e^{\pm i\beta x} = \cos\beta x \pm i \sin\beta x.$$

Then, $y = e^{\alpha x}\left[(C_1 + C_2)\cos\beta x + (C_1 i - C_2 i)\sin\beta x\right]$. $\qquad (g'')$

If C_1 and C_2 are unequal real numbers in (g''), then y has imaginary values.

However, y is real - valued if we choose C_1 and C_2 so that

$$\begin{aligned} iC_1 - iC_2 &= A, \\ C_1 + C_2 &= B \end{aligned} \qquad (g''')$$

where A and B are real. Observe that (g''') has a unique solution for C_1 and C_2 if A and B are assigned arbitrarily.

Then, (g') becomes

$$y = e^{\alpha x}(A \sin\beta x + B \cos\beta x). \qquad (g^{iv})$$

From (g^{iv}), we obtain the linear independent solutions $e^{\alpha x}\sin\beta x$ and $e^{\alpha x}\cos\beta x$, by use of (A=1, B=o) and (A=o, B=1), respectively.

If the characteristic equation $F(m) = o$ has imaginary roots

$m = \alpha \pm i\beta$, then $F(D)y=o$ has solutions $y= e^{\alpha x} \sin \beta x$ and $y = e^{\alpha x} \cos \beta x)$, and the general solution $(g^{iv}*)$.

We see at once that X 0 and T 0 is a solution to equations (14) and (15). This gives for the original problem of the vibrating string the trivial solution $y(x,t) \equiv 0$, which means that the string remains at rest. Hereafter, when we refer to a solution of our boundary problem we shall mean a solution not identically zero. To obtain the solution we have three cases to consider.

Case I.— $C > 0$. From elementary theory of differential equations we know that the most general solution of (14) is

$$X = Ae^{\lambda x} + Be^{-\lambda x} .$$

Applying the initial conditions, we obtain

$$X(0) = A + B = 0, \text{ whence } A = -B$$

$$X(L) = A(e^{\lambda L} - e^{-\lambda L}) = 0, \; A = 0 \quad (\lambda \neq 0).$$

Therefore, $A = B = 0$, and we have the trivial solution $X \equiv 0$.

Case II.— $C = 0$. The general solution of (14) is now

$$X = Ax + B.$$

But $X(o) = B=0$, and $X(L) = AL=0$, so that we have again the trivial solution $X \equiv 0$.

Case III. — $C < 0$. This will give the desired solution to our boundary - value problem.

From (14): $X'' + \lambda^2 X = o$

Auxiliary equation: $m^2 + \lambda^2 = o$

$$m = \pm \lambda i.$$

From (g): $X = e^{o} \left[D_1 \cos \lambda x + D_2 \sin \lambda x\right]$. (16)

From (15): $T'' + \lambda^2 a^2 T = o$

Auxiliary equation: $m^2 + \lambda^2 a^2 = o$

$$m = \pm a\lambda i.$$

From (g): $T = e^o \left[A_1 \cos a\lambda t + B_1 \sin a\lambda t\right].$ (17)

Substituting (16) and (17) in (10) yields

$$y(x,t) = (D_1 \cos\lambda x + D_2 \sin\lambda x)(A_1 \cos a\lambda t + B_1 \sin a\lambda t) \quad (18)$$

From (7): $y(o,t) = o.$

$$y(o,t) = D_1(A_1 \cos a\lambda t + B_1 \sin a\lambda t) = o.$$

Either $D_1 = o$ or $(A_1 \cos a\lambda t + B_1 \sin a\lambda t) = o.$

The solution would be trivial if the latter was o;

therefore, $D_1 = o.$

$$y(x,t) = D_2 \sin\lambda x(A_1 \cos a\lambda t + B_1 \sin a\lambda t) \quad (19)$$

$Y(x,t) = \sin\lambda x \ (B \cos a\lambda t + C \sin a\lambda t)$, where

$B = A_1/D_2$, $C = B_1/D_2$.

From (7) and (19): $y(L,t) = o.$

To satisfy the second condition in (7), we must have $\sin\lambda L = 0$,

$\lambda L = m\pi$, where m is an integer.

$$Y(L,t) = \sin L\lambda x(B \cos\lambda at + C \sin\lambda a t) = 0$$

$\sin L\lambda = o. \therefore$ we say $\lambda L = m\pi$, where m is an integer.

$$Y(x,t) = \sin \frac{m\pi}{L} x(B \cos \frac{m\pi}{L} at + C \sin \frac{m\pi}{L} at) \quad (20)$$

From (9) and (2): $Y_t(x,o) = o.$

$$\frac{\partial y}{\partial t}(x,t) = \sin \frac{m\pi x}{L}\left[-B\frac{m\pi a}{L} \sin \frac{m\pi}{L} at + C \frac{m\pi a}{L} \cos \frac{m\pi}{L} at\right]$$

$$Y_t(x,o) = \sin \frac{m\pi x}{L} C \frac{m\pi a}{L} = o.$$

One of the above factors must be zero.

$\sin \frac{m\pi}{L} x = o$	no velocity	No
$m = o$	no mass	No

$\pi = 0$ No

$a = 0$ No

$\frac{1}{L} = 0$ no length to the string No

$\therefore C = 0$

$$Y(x,t) = B \sin \frac{m\pi x}{L} \cos \frac{m\pi a t}{L} \qquad (21)$$

Equation (21) contains sine and cosine functions. Fourier, a French mathematician of the 19th Century, was led to a problem similar to our problem in his researches on heat. He solved the problem in the following manner.

Let us consider the series

$$F(x) = A + \sum_{k=1}^{\infty} a_k \cos \frac{k\pi x}{L} + \sum_{k=1}^{\infty} b_k \sin \frac{k\pi x}{L}, \quad \text{(h)}$$

, where $L = \frac{1}{2}$ the period, k is an integer,

1. $A = \frac{1}{2L}\int_{-L}^{L} F(x)dx,$
2. $b_k = \frac{1}{L}\int_{-L}^{L} F(x) \sin \frac{k\pi x}{L} dx$, and
3. $a_k = \frac{1}{L}\int_{-L}^{L} F(x) \cos \frac{k\pi x}{L} dx.$

Proof:

1. $A = \frac{1}{2L} \int_{-L}^{L} F(x)dx.$

Multiply (h) by dx and integrate from -L to L.

$$\int_{-L}^{L} F(x)dx = A\int_{-L}^{L} dx + \sum_{k=1}^{\infty} a_k \int_{-L}^{L} \cos \frac{k\pi}{L} xdx + \sum_{k=1}^{\infty} b_k \int_{-L}^{L} \sin \frac{k\pi x}{L} dx$$

$$\sum_{k=1}^{\infty} a_k \int_{-L}^{L} \cos \frac{k\pi x}{L} dx = \sum_{k=1}^{\infty} a_k \left(\frac{L}{k\pi}\right) \sin \frac{k\pi x}{L}\Big|_{-L}^{L} = 0 \;\forall k.$$

$$\sum_{k=1}^{\infty} b_k \int_{-L}^{L} \sin \frac{k\pi x}{L} dx = \sum_{k=1}^{\infty} b_k \, -\left(\frac{L}{k\pi}\right) \cos \frac{k\pi x}{L}\Big|_{-L}^{L} = 0 \;\forall k.$$

$$\int_{-L}^{L} F(x)\,dx = A\int_{-L}^{L} dx$$

$$= 2AL$$

$$\therefore A = \frac{1}{2L}\int_{-L}^{L} F(x)\,dx.$$

2. $b_k = \frac{1}{L}\int_{-L}^{L} F(x)\sin\frac{k\pi x}{L}\,dx.$

Proof:

Multiply (h) by $\sin\frac{m\pi x}{L}dx$ and integrate from $-L$ to L, where m is an integer.

$$\int_{-L}^{L} F(x)\sin\frac{m\pi x}{L}\,dx = A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx + \sum_{k=1}^{\infty} a_k\int_{-L}^{L}\cos\frac{k\pi x}{L}\sin\frac{m\pi x}{L}\,dx$$

$$+\sum_{k=1}^{\infty} b_k\int_{-L}^{L}\sin\frac{k\pi x}{L}\sin\frac{m\pi x}{L}\,dx$$

From 1: $A\int_{-L}^{L}\sin\frac{m\pi x}{L}dx = 0\ \forall m.$

By the sine of the sum and difference of two angles, we have

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y\\ \sin(x-y) &= \sin x\cos y - \cos x\sin y\end{aligned}\qquad (i)$$

$$\sin(x-y) - \sin(x+y) = -2\cos x\sin y$$

Let $x = \frac{k\pi x}{L}$; $y = \frac{m\pi x}{L}$.

$$\sum_{k=1}^{\infty} a_k\frac{1}{2}\int_{-L}^{L}\left[\sin\left(\frac{k\pi x}{L}+\frac{m\pi x}{L}\right) - \sin\left(\frac{k\pi x}{L}-\frac{m\pi x}{L}\right)\right]dx = 0 \quad \begin{matrix} m=k\\ m\neq k\end{matrix}$$

By the cosine of the sum and difference of two angles, we have

$$\begin{aligned}\cos(x+y) &= \cos x\cos y - \sin x\sin y)\\ \cos(x-y) &= \cos x\cos y + \sin x\sin y\end{aligned}\qquad (j)$$

$$\cos(x-y) - \cos(x+y) = 2\sin x\sin y$$

$$\sum_{k=1}^{\infty} b_k\frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{k\pi x}{L}-\frac{m\pi x}{L}\right) - \cos\left(\frac{k\pi x}{L}+\frac{m\pi x}{L}\right)\right]dx = b_k L\ \forall\ m=k$$

$$\int_{-L}^{L} F(x) \sin \frac{m\pi x}{L}\, dx = b_k L$$

$$\therefore\ b_k = \frac{1}{L}\int_{-L}^{L} F(x) \sin \frac{k\pi x}{L}\, dx\ .$$

3. $a_k = \frac{1}{L}\int_{-L}^{L} F(x) \cos \frac{k\pi x}{L}\, dx.$

Proof:

Multiply (h) by $\cos \frac{m\pi x}{L}\, dx$ and integrate from -L to L.

$$\int_{-L}^{L} F(x) \cos \frac{m\pi x}{L}\, dx = A\int_{-L}^{L} \cos \frac{m\pi x}{L}\, dx + \sum_{k=1}^{\infty} a_k \int_{-L}^{L} \cos \frac{k\pi x}{L} \cos \frac{m\pi x}{L}\, dx +$$

$$\sum_{k=1}^{\infty} b_k \int_{-L}^{L} \sin \frac{k\pi x}{L} \cos \frac{m\pi x}{L}\, dx$$

From 1: $A\int_{-L}^{L} \cos \frac{m\pi x}{L}\, dx = 0\ \forall m.$

From (j): $\sum_{k=1}^{\infty} a_k \frac{1}{2}\int_{-L}^{L}\left[\cos\left(\frac{k\pi x}{L} + \frac{m\pi x}{L}\right) + \cos\left(\frac{k\pi x}{L} - \frac{m\pi x}{L}\right)\right] dx = La_k \quad m=k$

From 2: $\sum_{k=1}^{\infty} b_k \int_{-L}^{L} \sin \frac{k\pi x}{L} \cos \frac{k\pi x}{L}\, dx = 0 \quad m = k \text{ and } m \neq k.$

$$\therefore\ a_k = \frac{1}{L}\int_{-L}^{L} F(x) \cos \frac{k\pi x}{L}\, dx\ .$$

To satisfy the last boundary condition of (8) we use the principle of superposition and write (21) formally

$$Y(x,t) = b_1 \sin \frac{\pi x}{L} \cos \frac{\pi a t}{L} + b_2 \sin \frac{2\pi x}{L} \cos \frac{2 a\pi t}{L} + \cdots$$
$$+ b_k \sin \frac{k\pi x}{L} \cos \frac{k\pi a t}{L}\ .$$

$$Y(x,0) = b_1 \sin \frac{\pi x}{L} + b_2 \sin \frac{2\pi x}{L} + \cdots b_k \sin \frac{k\pi x}{L}\ .$$

Hence from (H) we have

$$b_k = \frac{1}{L}\int_{-L}^{L} Y(x,0) \sin \frac{k\pi x}{L}\, dx.$$

Using (8) and integrating yields

$$b_k = \frac{2}{L}\left[\int_{0}^{\frac{L}{2}} \frac{2hx}{L} \sin \frac{k\pi x}{L}\, dx + \int_{\frac{L}{2}}^{L} \frac{2h}{L}(L-x) \sin \frac{k\pi x}{L}\, dx\right]$$

$$b_k = \frac{8\,h}{k^2\pi^2} \sin \frac{k\pi}{2} .$$

The described motion is

$$Y(x,t) = \frac{8h}{\pi^2}\left(\sin \frac{\pi x}{L} \cos\frac{\pi a t}{L} - \frac{1}{9} \sin \frac{3\pi x}{L} \cos \frac{3\pi a t}{L} + \cdots\right). \quad (22)$$

CHAPTER V

SUMMARY

There is a physical significance to the various terms of (22). Each term represents a particular mode in which the system vibrates. The first term, apart from the constant factor is $\sin \frac{\pi x}{L} \cos \frac{\pi a t}{L}$, which represents the first mode of vibration. At $t = o$ the graph of the string is shown in Figure 5, where the vertical scale has been enlarged. As t varies, the tendency is for the string in this first mode of vibration to oscillate about the equilibrium position (x axis) with frequency (determined from $\cos \frac{\pi a t}{L}$) given by $\frac{a}{2L}$ cycles per second. This lowest frequency is called the fundamental frequency or first harmonic. Since $b_2 = o$ in (22), the

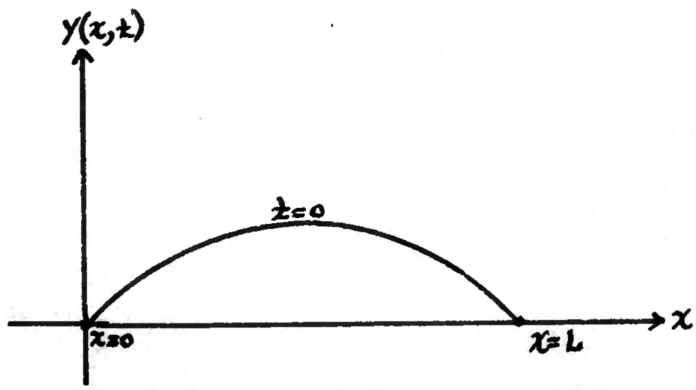

Figure 5

term corresponding to the second mode of vibration is missing. The third mode present has a frequency of $\frac{3a}{2L}$ cycles per second, which is called the harmonic (or sometimes the second overtone).[1] Similarly the fifth harmonic frequency is $\frac{5a}{2L}$ cycles per second.

It is found that in the case of the vibrating string all the harmonic frequencies are integer multiples of the fundamental frenquency. Thus 3, 5, ------ are integer multiples of $\frac{a}{2L}$. Whenever this occurs we say that we have music, as for example, in our violin string. In the string of our problem the sound produced is of such very low frequency that it is not in the audible range. As we increase the tension we increase the frequency and the result is a musical tone. The various coefficients of the terms in (22) have physical significance. These coefficients measure the intensity of the various modes. The higher frequencies correspond to lower intensities.

In more advanced applied mathematics, it is found that a vibrating circular drumhead also has various modes of vibration and corresponding frequencies. However, these frequencies are not multiples of a fundamental vibration and as a consequence noise is heard instead of music.

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